Spherical functions on compact Gelfand pairs of rank one

Maarten van Pruijssen

(jt. with G. Heckman)

Universität Paderborn

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Table of contents

Matrix valued polynomials

Fix $N \in \mathbb{N}$ and denote $\mathbb{M} = \mathrm{End}(\mathbb{C}^N)$. Let $(\cdot)^*$ denote Hermitian adjoint.

- A **matrix valued polynomial** is an element $P \in \mathbb{M}[x]$.
- A **matrix weight** on an interval $I \subset \mathbb{R}$ is a map $W : I \to \mathbb{M}$ with $W(x)^* = W(x)$ and $W(x) > 0$ almost everywhere and

$$\int_I x^n W(x) dx \in \mathbb{M} \quad \text{(finite moments)}.$$

- Define the pairing

$$\langle P, Q \rangle_W = \int_I P(x)^* W(x) Q(x) dx \in \mathbb{M}$$

for $P, Q \in \mathbb{M}[x]$.

Matrix valued orthogonal polynomials

The pairing $\langle \cdot, \cdot \rangle_W$ is a **matrix valued inner product**, i.e. it has the following properties:

- $\langle P, Q \rangle_W^* = \langle Q, P \rangle_W$,
- $\langle P, QA \rangle_W = \langle P, Q \rangle_W A$,
- $\langle P, P \rangle_W \geq 0$ and $\langle P, P \rangle_W = 0$ iff $P = 0$

for all $P, Q \in \mathbb{M}[x]$ and all $A \in \mathbb{M}$.

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for all $P, Q \in \mathbb{M}[x]$ and all $A \in \mathbb{M}$.

A family $\{P_n : n \in \mathbb{N}\}$ with $P_n \in \mathbb{M}[x]$ is called a **family of matrix valued orthogonal polynomials** (MVOPs) if $\deg(P_n) = n$, if the coefficient of x^n is invertible and if $\langle P_n, P_m \rangle_W = C_n \delta_{nm}$.

Matrix valued differential operators

Fix a matrix weight W and a family of MVOPs $\{P_n : n \in \mathbb{N}\}$. Let $D \in \mathbb{M}[x] \otimes \mathbb{C}[\partial_x]$ and suppose that

- $DP_n = P_n \Lambda_n$ for all n,
- D is symmetric, i.e. $\langle DP, Q \rangle_W = \langle P, DQ \rangle_W$ for all $P, Q \in \mathbb{M}[x]$.

Then (W, D) is called a matrix valued classical pair (MVCP).

Examples

$N = 1$, $\mathbb{M} = \mathbb{C}$. The classical orthogonal polynomials of

- Hermite, $w(x) = e^{-x^2}$ on $\mathbb{R}$,
- Laguerre, $w(x) = x^\alpha e^{-x}$ on $\mathbb{R}_{\geq 0}$,
- Jacobi, $w(x) = (1-x)^\alpha (1+x)^\beta$ on $[-1, 1]$

are eigenfunctions of a second order differential equation. In fact, this property characterizes them (Bochner).

Examples

Matrix valued classical pairs related to group theory

- $(\mathrm{SU}(3), \mathrm{U}(2))$ by Grünbaum, Pacharoni and Tirao,
- $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$ by Koelink, van Pruijssen and Román,
- $(\mathrm{SO}(4), \mathrm{SO}(3))$ by Tirao and Zurrián,
- ...
- Almost all compact Gel'fand pairs (G, K) of rank one with appropriate K-type, by Heckman and van Pruijssen.

Goal of this lecture

Goal

Discuss

- *a spectral problem for compact Lie groups,*
- *classification of the solutions,*
- *construction of matrix valued classical pairs (W, D) from group theory.*

No multiplicities

Let G be a compact connected Lie group, $K \subset G$ a closed connected subgroup.

- Identify $\widehat{G} \longleftrightarrow P_G^+$ via $\pi_\lambda \longleftrightarrow \lambda$.
- Identify $\widehat{K} \longleftrightarrow P_K^+$ via $\tau_\mu \longleftrightarrow \mu$.
- For $\lambda \in P_G^+$ and $\mu \in P_K^+$ define $m_\lambda^{G,K}(\mu) = [\pi_\lambda|_K : \tau_\mu]$

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- For $\lambda \in P_G^+$ and $\mu \in P_K^+$ define $m_\lambda^{G,K}(\mu) = [\pi_\lambda|_K : \tau_\mu]$

Definition

- A **multiplicity free triple** (MFT): (G, K, μ) with a weight $\mu \in P_K^+$ such that for all $\lambda \in P_G^+$ we have $m_\lambda^{G,K}(\mu) \leq 1$.
- A **multiplicity free system** (MFS): (G, K, F) with a face $F \subset P_K^+$ such that for all $\mu \in F$, (G, K, μ) is a multiplicity free triple.

Examples

Recall: a **compact Gelfand pair** is a pair of compact groups (G,K) with $K \subset G$ for which $m_\lambda^{G,K}(0) \leq 1$ for all $\lambda \in P_G^+$.
Examples of MFSs:

- (G,K,F) with (G,K) a compact Gelfand pair and $F = \{0\}$,
- $(\mathrm{SU}(n+1), \mathrm{U}(n), P_{\mathrm{U}(n)}^+)$,
- $(\mathrm{G}_2, \mathrm{SU}(3), F)$ with $F = \mathbb{N}\omega_1$ or $F = \mathbb{N}\omega_2$,
- ...

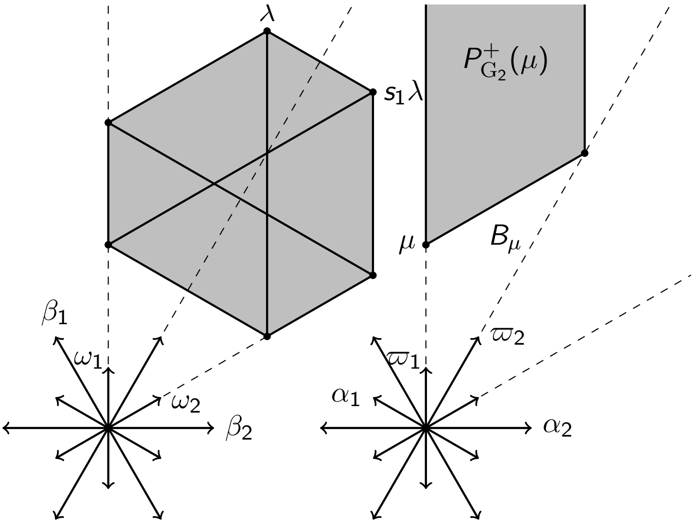

Figure : Branching from G_2 to $\mathrm{SU}(3)$ on the left and the μ-well on the right.

Classification

Theorem

A MFS (G, K, F) with (G, K) a Gelfand pair of rank one is isogenous to one of the following.

G	K	λ_{sph}	*faces* F
$\mathrm{SU}(n+1)$	$\mathrm{U}(n)$	$\varpi_1 + \varpi_n$	*any*
$\mathrm{SO}(2n)$	$\mathrm{SO}(2n-1)$	ϵ_1	*any*
$\mathrm{SO}(2n+1)$	$\mathrm{SO}(2n)$	ϖ_1	*any*
$\mathrm{Sp}(2n)$	$\mathrm{Sp}(2n-2) \times \mathrm{Sp}(2)$	ϖ_2	$\dim F \leq 2$
F_4	$\mathrm{Spin}(9)$	ϖ_1	$\dim F \leq 1$ *or* $F = \langle \omega_1, \omega_2 \rangle$
$\mathrm{Spin}(7)$	G_2	ϖ_3	$\dim F \leq 1$
G_2	$\mathrm{SU}(3)$	ϖ_1	$\dim F \leq 1$

Parts of the proof

- (Classical) branching theorems,
- theory of spherical varieties.

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Fact

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- $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{n}^+$.

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Fact

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Theorem

Define $M = Z_K(A)$. *The triple* (G, K, F) *is a MFS if and only if for each* $\tau \in \widehat{K}$ *of highest weight* $\mu \in F$, *the restriction* $\tau|_M$ *decomposes multiplicity free.*

That twisted embedding!

Consider $(\mathrm{F}_4, \mathrm{Spin}(9))$. Then $M \cong \mathrm{Spin}(7)$ and

$$\mathrm{Spin}(7) \to \mathrm{Spin}(8) \to \mathrm{Spin}(9)$$

via *twisted* embedding $\mathrm{Spin}(8) \to \mathrm{Spin}(9)$.

- The branching rules from $\mathrm{Spin}(9)$ to $\mathrm{Spin}(7)$ are **not** classical.
- The faces $\langle \omega_1, \omega_2 \rangle$, $\mathbb{N}\omega_3$ and $\mathbb{N}\omega_4$ give MFSs.
- The other faces are excluded by dimension.

Classification

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G	K	λ_{sph}	*faces* F
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$\mathrm{Spin}(7)$	G_2	ϖ_3	$\dim F \leq 1$
G_2	$\mathrm{SU}(3)$	ϖ_1	$\dim F \leq 1$

Remember this well!

Definition

Let (G, K, μ) be a multiplicity free triple. Define

$$P_G^+(\mu) = \{\lambda \in P_G^+ | m_\lambda^{G,K}(\mu) = 1\}.$$

The set $P_G^+(\mu)$ is called the μ-well.

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Fact
Let (G, K) be a Gelfand pair of rank one. Then $P_G^+(0) = \mathbb{N}\lambda_{\mathrm{sph}}$, where λ_{sph} is the minimal spherical weight.

Remember this well!

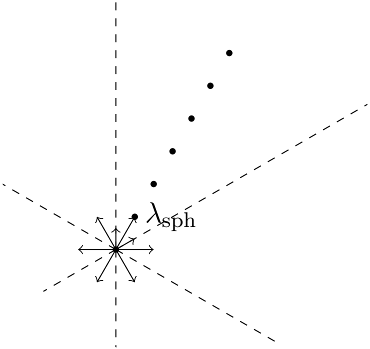

Figure : The 0-well for $(\mathrm{SU}(3), \mathrm{U}(2))$.

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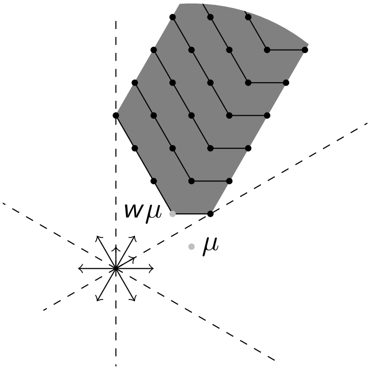

Fact

$$P_G^+(\mu) = B_\mu + \mathbb{N}\lambda_{\mathrm{sph}}.$$

On the shape of the well

- If $\lambda \in P_G^+(\mu)$ then $\lambda + \lambda_{\mathrm{sph}} \in P_G^+(\mu)$.
- Recall $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{n}^+$ and $M = Z_K(A)$.

Theorem

Let $\lambda \in P_G^+(\mu)$. Then M acts irreducibly on $V_\lambda^{\mathfrak{n}^+}$.

- If $\lambda \in P_G^+(\mu)$ then the type of M on $V_\lambda^{\mathfrak{n}^+}$ is $\nu \in P_M^+(\mu)$.
- The map $P_G^+(\mu) \to P_M^+(\mu)$ is surjective.

It follows that the bottom has $|B_\mu| = |P_M^+(\mu)| = \dim \mathrm{End}_M(V_\mu)$ elements.

The space of μ-spherical functions

- Fix a multiplicity free system (G, K, F) with (G, K) of rank one and G other than F_4.
- Fix $\mu \in F$ and let $\tau : K \to \mathrm{GL}(V_\mu)$ denote the corresponding irreducible representation.
- Let $R(G)$ denote the (convolution) algebra of matrix coefficients of G and define the $K \times K$-action on $R(G) \otimes \mathrm{End}(V_\mu)$ by

$$(k_1, k_2)(m \otimes Y)(g) = m(k_1^{-1} g k_2) \otimes \tau(k_1) Y \tau(k_2)^{-1}.$$

The space of μ-spherical functions

Definition

The space $E^{\mu} = (R(G) \otimes \mathrm{End}(V_{\mu}))^{K \times K}$ is called the **space of μ-spherical functions**.

Note that $\Phi \in E^{\mu}$ satisfies

$$\Phi(k_1 g k_2) = \tau(k_1)\Phi(g)\tau(k_2) \quad \forall k_1, k_2 \in K, g \in G.$$

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Fact

- *E^μ is a free, finitely generated E^0-module.*

A basis for the space of μ-spherical functions

Let $\lambda \in P_G^+(\mu)$ and let $\pi_\lambda : G \to \mathrm{GL}(V_\lambda)$ denote the corresponding representation space. Then

- $V_\lambda = V_\mu \oplus V_\mu^\perp$ and
- we denote by $b : V_\mu \to V_\lambda$ a unitary K-equivariant embedding and by $b^* : V_\lambda \to V_\mu$ its Hermitian adjoint.

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Definition

The elementary spherical function of type μ associated to $\lambda \in P_G^+(\mu)$ is defined by

$$\Phi_\lambda^\mu : G \to \mathrm{End}(V_\mu) : g \mapsto b^* \circ \pi_\lambda(g) \circ b.$$

A basis for the space of μ-spherical functions

Definition

Let $\Phi_1, \Phi_2 \in E^\mu$. Define

$$\langle \Phi_1, \Phi_2 \rangle_{\mu,G} = \int_G \mathrm{tr}\left(\Phi_1(g)^* \Phi_2(g)\right) dg$$

with dg the normalized Haar measure.

A basis for the space of μ-spherical functions

Theorem

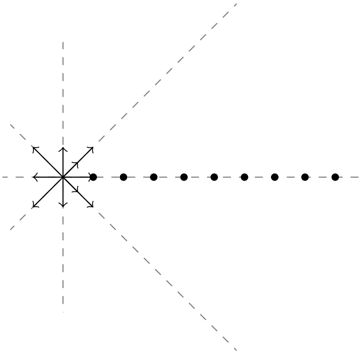
- *The pairing* $\langle \cdot, \cdot \rangle_{\mu,G} : E^\mu \times E^\mu \to \mathbb{C}$ *is a Hermitian inner product and*
$$\langle \Phi^\mu_\lambda, \Phi^\mu_{\lambda'} \rangle_{\mu,G} = c_\lambda \delta_{\lambda,\lambda'}.$$
- $\{\Phi^\mu_\lambda : \lambda \in P^+_G(\mu)\}$ *is a basis of* E^μ.

Recall that for $\mu = 0$ we have $P^+_G(0) = \lambda_{\mathrm{sph}}\mathbb{N}$. Denote

$$\phi = \Phi^0_{\lambda_{\mathrm{sph}}}.$$

We have $E^0 = \mathbb{C}[\phi]$, i.e. E^0 is a ring. Note that $\phi(k_1 g k_2) = \phi(g)$ for all $k_1, k_2 \in K$ and all $g \in G$.

Remember this well



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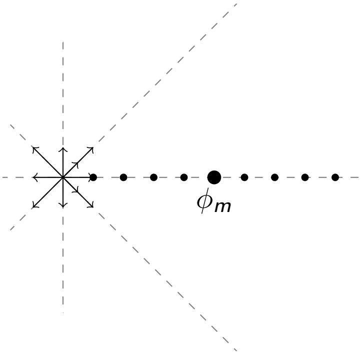

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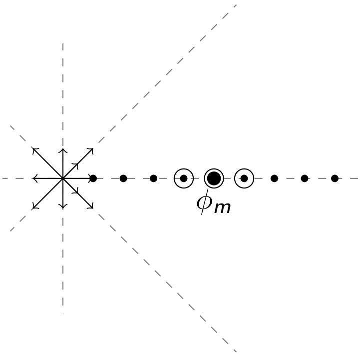
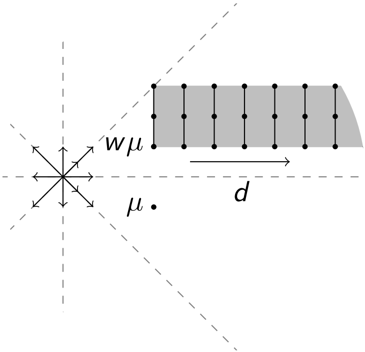


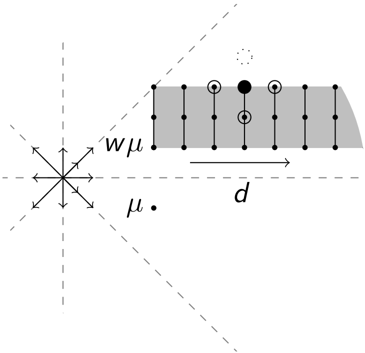
$$\phi\phi_m = a_m\phi_{m+1} + b_m\phi_m + c_m\phi_{m-1}.$$

Remember this well



Every dot in the picture represents a basis vector of E^{μ}. The bottom B_{μ} consists of degree zero basis elements.

Remember this well



$$\phi \Phi_{d,\nu} = a_{d,\nu}\Phi_{d+1,\nu} + \Big(\sum_{\nu' \in B_\mu} b_{d,\nu}^{\nu'} \Phi_{d,\nu'}\Big) + c_{d,\nu}\Phi_{d-1,\nu}$$

Differential properties

- The Casimir operator $\Omega \in U(\mathfrak{g}_{\mathbb{C}})$ of G gives a differential operator $\Omega_\mu : E^\mu \to E^\mu$,
- Every elementary spherical function Φ^μ_λ is an eigenfunction of Ω_μ with scalar eigenvalue.

Restriction to a circle

Recall $G = KAK$ with $A \cong S^1$.

Fact

- $\Phi^\mu(A) \subset \mathrm{End}_M(V_\mu)$ *for* $\Phi^\mu \in E^\mu$,
- $\dim \mathrm{End}_M(V_\mu) = |B_\mu|$,
- $E^\mu \cong \mathbb{C}[\phi] \otimes \mathbb{C}^{|B_\mu|}$.

We will make the isomorphism $E^\mu \cong \mathbb{C}[\phi] \otimes \mathbb{C}^{|B_\mu|}$ explicit.

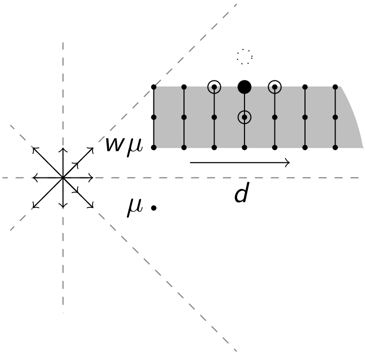
Stick together

- Denote $E^\mu_A = \{\Phi|_A : \Phi \in E^\mu\}$,
- In view of $\mathrm{End}_M(V_\mu) \cong \mathbb{C}^{|B_\mu|}$, denote $\Phi|_A \leftrightarrow \Psi$.
- The restricted elementary spherical functions correspond to $\Psi_{d,\nu}$
- For each degree d we collect the $|B_\mu|$ basis elements of degree d and arrange them in a matrix to obtain

$$\Psi_d : A \to \mathrm{End}(\mathbb{C}^{|B_\mu|})$$

Note that Ψ_d is a square matrix.

Stick together



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Stick together

- The functions $\{\Psi_d : d \in \mathbb{N}\}$ satisfy a three term recurrence relation:

$$\phi\Psi_d = \Psi_{d+1}A_d + \Psi_d B_d + \Psi_{d-1}C_d,$$

  with $A_d, B_d, C_d \in \mathrm{End}(\mathbb{C}^{|B_\mu|})$ and A_d invertible.
- The functions Ψ_d are eigenfunctions of the operator $\mathrm{rad}(\Omega_\mu)$,

$$\mathrm{rad}(\Omega_\mu)\Psi_d = \Psi_d\Lambda_d,$$

  with $\Lambda_d \in \mathrm{End}(\mathbb{C}^{|B_\mu|})$ diagonal.

A base change

- The map
$$\bigoplus_{i=1}^{|B_\mu|} E_A^\mu \to \mathbb{C}[\phi] \otimes \mathrm{End}(\mathbb{C}^{|B_\mu|})$$
is given by multiplication by Ψ_0^{-1}, because of the three-term recurrence relation.
- Denote $Q_d = \Psi_0^{-1}\Psi_d$.

The importance of being Ψ_0

- The corresponding polynomials $Q_d \in \mathbb{C}[\phi] \otimes \mathrm{End}(\mathbb{C}^{d(\tau)})$ constitute a family of MVOPs.
- The weight is $W_\mu(\phi)d\phi$ upon integrating over $\phi(S^1)$, with

$$W_\mu(\phi)(s) = \Psi_0(s)^*\Psi_0(s)(1-\phi(s))^\alpha(1+\phi(s))^\beta.$$

- Conjugating $\mathrm{rad}(\Omega_\mu)$ with Ψ_0 gives a symmetric differential operator

$$D_\mu \in \mathbb{C}[\phi] \otimes \mathrm{End}(\mathbb{C}^{d(\mu)}) \otimes \mathbb{C}[\partial_\phi]$$

  of order two for which the Q_d are eigenfunctions with eigenvalues $\Lambda_d \in \mathrm{End}(\mathbb{C}^d)$ (diagonal).

The importance of being Ψ_0

- The three-term recurrence relation

$$\phi \Psi_d = \Psi_{d+1} A_d + \Psi_d B_d + \Psi_{d-1} C_d$$

multiplied on the left by Ψ_0^{-1} gives the three-term recurrence relation

$$\phi Q(\phi)_d = Q_{d+1}(\phi) A_d + Q_d(\phi) B_d + Q_{d-1}(\phi) C_d$$

Invertibility of Ψ_0

Theorem

$\Psi_0(s)$ is invertible if $\phi|_{S^1}$ is immersive in $s \in S^1$.

Sketch of proof.

- $\Psi_0|_{S^1}$ and $\phi|_{S^1}$ can be extended to functions on $\mathbb{C}^\times$.
- We show that on $\mathbb{C}^\times$ the function Ψ_0 satisfies a first order differential equation

$$z^2 \partial_z \phi(z) \partial_z \Psi_0(z) = \Psi_0(z)(R\phi(z) + S)$$

with $R, S \in \mathrm{End}(\mathbb{C}^{d(\tau)})$.

Invertibility of Ψ_0

(i) $\phi\Psi_d = \Psi_{d+1}A_d + \Psi_d B_d + \Psi_{d-1}C_d$,

(ii) $D_{\mu_0}\phi = \lambda\phi$,

(iii) $D_\mu\Psi_d = \Psi_d\Lambda_d$.

where $D_\mu = (z\partial_z)^2 + c(z)z\partial_z + F_\mu(z)$ with $F_\mu(z)$ matrix valued, $F_0(z) = 0$.

Calculate

$$(z\partial_z)^2(\phi(z)\Psi_0(z)) = (z\partial_z)^2(\Psi_1(z)A_0 + \Psi_0(z)B_0)$$

and get rid of second order derivatives and Ψ_1 using (i), (ii) and (iii).

Invertibility of Ψ_0

We obtain

$$2z^2\phi'(z)\Psi_0'(z) = \\ \Psi_0(z)\{\{1-\Lambda_0-\lambda\}\phi(z)+\Lambda_0 B_0 - B_0 A_0^{-1}\Lambda_1 A_0\}.$$

QED.

Invertibility of Ψ_0

We obtain

$$2z^2\phi'(z)\Psi_0'(z) = \Psi_0(z)\{\{1-\Lambda_0-\lambda\}\phi(z)+\Lambda_0 B_0 - B_0 A_0^{-1}\Lambda_1 A_0\}.$$

QED.

Corollary

Upon changing the variable $x=\phi(z)$ we have

$$(D_\mu)_{\phi=x} = (r_2x^2+r_1x+r_0)\partial_x^2 P(x)+\{\lambda+(Rx+S)\}\partial_x P(x)+\Lambda_0 P(x).$$

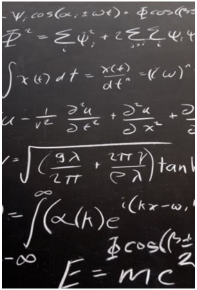
with $(r_2\phi^2+r_1\phi+r_0)=(z\phi')^2$.

Conclusion

- The spherical functions of type μ give a MVCP (W_μ, D_μ),
- For $\mu = 0$ the construction yields Jacobi polynomials,
- The construction applied to $(\mathrm{SU}(3), \mathrm{U}(2))$ gives the family of MVOPs found by Grünbaum et al.
- Explicit knowledge of Ψ_0 yields an expression of the matrix weight and a symmetric second order differential operator.

Outlook

- Implementation of Ψ_0-calculations in *GAP* (jt. with Pablo Román).
- Show that the weight matrices are not diagonalizable by constant matrix,
- Find general approach to analyze the μ-well (using symplectic geometry, moment maps),
- Work out the well in the case $(\mathrm{F}_4, \mathrm{Spin}(9))$,
- Classify all multiplicity free systems (use the classification of compact Gel'fand pairs),
- Analyze the module structure and the structure of the μ-well using general methods (moment maps, convexity, ...)
- ...



Thank you!